Point Modules over a $\mathbb{Z}^2$-graded quantum $\mathbb{P}^3$

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Non-commutative algebraic geometry

THE FUNDAMENTAL IDEA:

a non-commutative algebraic variety/scheme is defined by saying what its category of "quasi-coherent sheaves" (or modules) is.

Setup

- $\Gamma :=$ finitely generated abelian group

- $A :=$ Γ-graded ring

- $\mathfrak{a} :=$ Γ-graded ideal in A, the *irrelevant* ideal

- $\mathsf{Gr}(A, \Gamma) :=$ the category of Γ-graded, right A-modules with morphisms

$$\{f \in \mathsf{Hom}_A(M, N) \mid f(M_i) \subset N_i \text{ for all } i \in \Gamma\}$$

- $M \in \mathsf{Gr}(A, \Gamma)$ is $\mathfrak{a}$-*torsion* if for each $m \in M$, $m\mathfrak{a}^n = 0$ for $n >> 0$.

- $\mathfrak{a}\mathsf{Tors} :=$ the full subcategory of $\mathsf{Gr}(A, \Gamma)$ of $\mathfrak{a}$-torsion modules

• $X := \mathsf{Proj}_{\mathsf{nc}}(A, \Gamma, \mathfrak{a})$ is the non-commutative scheme defined by

$$\mathsf{Qcoh} X := \frac{\mathsf{Gr}(A, \Gamma)}{\mathfrak{a}\mathsf{Tors}}$$

• $\pi^* :=$ the quotient functor $\mathsf{Gr}(A, \Gamma) \to \mathsf{Qcoh} X$

Motivation from toric varieties

- work over $\mathbb{C}$

Definition. *A toric variety is an irreducible variety X such that:*

(i) $(\mathbb{C}^*)^n$ is a Zariski open subset of X

(ii) the action of $(\mathbb{C}^*)^n$ on itself extends to an action of $(\mathbb{C}^*)^n$ on X.

Proposition. *Let X be a toric variety. Then for some positive integers r and n, there is a linear action of $(\mathbb{C}^*)^r$ on $\mathbb{C}^n$ and a closed $(\mathbb{C}^*)^r$-stable subvariety $Z \subset \mathbb{C}^n$ such that X is isomorphic to the categorical quotient $(\mathbb{C}^n - Z)/(\mathbb{C}^*)^r$.*

- $X := (\mathbb{C}^n - Z)/(\mathbb{C}^*)^r$ as in the proposition

- $\Gamma := \mathrm{Hom}_{\mathbb{Z}}((\mathbb{C}^*)^r, \mathbb{C}^*) \cong \mathbb{Z}^r$

- $R := \mathbb{C}[x_1, \ldots x_n]$ graded by Γ

- $\mathfrak{a} :=$ the Γ-graded ideal in R defining Z

- The action of $(\mathbb{C}^*)^r$ on $\mathbb{C}^n$ is equivalent to a Γ-grading on R.

e.g. $\mathbb{C}^* \times \mathbb{C}^*$ acting on $\mathbb{C}^4$ via

$$(\alpha, \beta).(d, u, x, y) = (\alpha d, \beta^2 u, \alpha^3 \beta^{-2} x, \alpha\beta y)$$

is equivalent to the $\mathbb{Z}^2$-grading on $\mathbb{C}[d, u, x, y]$ given by $\deg(d) = (1, 0)$, $\deg(u) = (0, 2)$, $\deg(x) = (3, -2)$, $\deg(y) = (1, 1)$.

Theorem (Cox, 1995). *If X is a smooth toric variety then*

$$\mathit{Qcoh} X \equiv \frac{\mathit{Gr}(R, \Gamma)}{\mathfrak{a}\,\mathit{Tors}}.$$

In other words, when X is a smooth toric variety,

$$X \cong \mathsf{Proj}_{\mathsf{nc}}(R, \Gamma, \mathfrak{a}).$$

A quantum $\mathbb{P}^3$ example

- Let $r \in \mathbb{C}^*$. Let B be the down-up algebra $\mathbb{C}\langle x, y\rangle/J$ where J is generated by the cubic elements

$$x^2y-(r+r^{-1})xyx+yx^2, y^2x-(r+r^{-1})yxy+xy^2.$$

- Let $r, \alpha \in \mathbb{C}^*$. Let $A = \mathbb{C}\langle d, u, x, y\rangle/I$ where I is the ideal generated by the six quadratic elements

$$xd - dy, \quad yd - \alpha dx, \quad ux - ryu, \quad uy - \alpha r^{-1}xu,$$

$$du - (xy - ryx), \qquad ud + \alpha r(xy - r^{-1}yx).$$

- A is a generalized Laurent polynomial ring over the down-up algebra B.

- A is a quantum $\mathbb{P}^3$.

• If r is not a root of unity and α is a root of unity, then A is not finite over its center. Furthermore the point-scheme automorphism of A has finite order.

• For the rest of the talk, $r \in \mathbb{C}^*$ is not a root of unity.

- Grade A by defining $\deg(d) = (2,0)$, $\deg(u) = (0,2)$, and $\deg(x) = \deg(y) = (1,1)$.

- $\Gamma := \mathbb{Z}^2$

- A is a Γ-graded algebra.

- $T :=$ the torus $\mathbb{C}^* \times \mathbb{C}^*/\{(1,1),(-1,-1)\}$

- T acts on $\mathbb{C}^4$ via:

$$(\alpha,\beta).(d,u,x,y) = (\alpha^2 d, \beta^2 u, \alpha\beta x, \alpha\beta y)$$

- $Z := \{p \in \mathbb{C}^4 \mid \mathrm{Stab}_T(p) \text{ is nontrivial } \}$

- $Z = Z(\mathfrak{a})$ where $\mathfrak{a} = (d,u)(x,y)$

- $X := \mathrm{Proj}_{\mathrm{nc}}(A,\Gamma,\mathfrak{a})$

• $R :=$ polynomial ring $\mathbb{C}[d,u,x,y]$, with the same Γ-grading as A

• $\mathcal{O}(p) :=$ the T-orbit of p for $p \in \mathbb{C}^4$

• $I(p) :=$ the ideal of R vanishing on the closure of $\mathcal{O}(p)$

• Cox's theorem implies that

$$\{\text{points in } \mathsf{Proj}_{\mathsf{nc}}(R,\Gamma,\mathfrak{a})\} \longleftrightarrow \{\mathcal{O}(p) \mid p \in \mathbb{C}^4 - Z(\mathfrak{a})\},$$

so it is appropriate to call $R/I(p)$ a *point module* for R.

• Let $p \in \mathbb{C}^4 - Z(\mathfrak{a})$.

Case 1: p has exactly two nonzero coordinates.

Then $I(p)$ is generated by one of the pairs $(d, x), (d, y), (u, y), (u, x)$.

Case 2: p has exactly three nonzero coordinates.

Then $I(p)$ is generated by one of the pairs $(x, \mu y^2 - \nu ud), (y, \mu x^2 - \nu ud)$ or

$(u, \lambda y - \gamma x), (d, \lambda y - \gamma x)$, where $\mu, \nu, \lambda, \gamma \in \mathbb{C}^*$.

Case 3: All coordinates of p are nonzero.

Then $I(P)$ is generated by $(\lambda x^2 - \gamma ud, \lambda y^2 - \epsilon ud)$ for some $\lambda, \gamma, \epsilon \in \mathbb{C}^*$.

- An object $\mathcal{O} \in \mathsf{Qcoh}X$ is *simple* if $\mathcal{O} = \pi^*(P)$ for some $P \in \mathsf{Gr}(A, \Gamma)$ such that P is not $\mathfrak{a}$-torsion and for any nonzero submodule Q of P, the quotient P/Q is $\mathfrak{a}$-torsion.

- A *point* of $X = \mathsf{Proj}_{\mathsf{nc}}(A, \Gamma, \mathfrak{a})$ is a simple object of $\mathsf{Qcoh}X$.

- When do the modules $A/I(p)A$ represent points of X?

Theorem. *Let* $X = \mathit{Proj}_{nc}(A, \Gamma, \mathfrak{a})$ *and let* $\pi^* : \mathit{Gr}(A, \Gamma) \to \mathit{Qcoh}X$ *be the quotient functor.*

1. *If* I *is the right ideal in* A *generated by one of the following pairs of elements,*

   $$(u, \lambda y - \gamma x),\ (d, \lambda y - \gamma x), \quad (\lambda, \gamma) \in \mathbb{P}^1,$$

   then $\pi^*(A/I)$ *is a point of* X*.*

2. *If* I *is the right ideal in* A *generated by one of the following pairs of elements,*

   $$(x, \mu y^2 - \nu ud),\ (y, \mu x^2 - \nu ud),\ (\mu, \nu) \in \mathbb{P}^1 - \{0\},$$

   $$(x, \mu y^2 - \nu du),\ (y, \mu x^2 - \nu du),\ (\mu, \nu) \in \mathbb{P}^1 - \{0\},$$

   then $\pi^*(A/I) = 0$ *in* $\mathit{Qcoh}X$*.*

• The Hilbert series of $A/(u, \lambda y - \gamma x)A$ and $A/(d, \lambda y - \gamma x)A$ are respectively,

$$\frac{1}{(1-s^2)(1-st)} \quad \text{and} \quad \frac{1}{(1-t^2)(1-st)}.$$

• In part 2 of the theorem, when $(\mu, \nu) = 0 \in \mathbb{P}^1$, each of the corresponding modules fit into two exact sequences.
For example:

$$0 \to \frac{A}{(y,d)A}(0,-2) \xrightarrow{u\cdot} \frac{A}{(x,ud)A} \to \frac{A}{(x,u)A} \to 0$$

$$0 \to \frac{A}{(y,u)A}(-2,0) \xrightarrow{d\cdot} \frac{A}{(x,ud)A} \to \frac{A}{(x,d)A} \to 0$$

Some case 3 modules

- $M_{(\lambda,\gamma,\epsilon)} := A/(\lambda x^2 - \gamma ud, \lambda y^2 - \epsilon ud)A$,
$$(\lambda,\gamma,\epsilon) \in \mathbb{P}^2$$

- $\beta := \frac{r}{(r^2-1)^2}$

- $F_{r,\alpha}(X,Y,Z) := \beta X^2 - \alpha^2 YZ$ in $\mathbb{C}[X,Y,Z]$

- $C :=$ the conic in $\mathbb{P}^2$ defined by $F_{r,\alpha}(X,Y,Z)$

Theorem. *Let $(\lambda,\gamma,\epsilon) \in \mathbb{P}^2$ with $\lambda \neq 0$. Then $\pi^*(M_{(\lambda,\gamma,\epsilon)})$ is a point of X if and only if $(\lambda,\gamma,\epsilon)$ lies on C.*

- If $\lambda = 0$, then $\pi^*(M_{(0,\gamma,\epsilon)}) \neq 0$ and is not a simple object of Qcoh$X$.

- A similar theorem holds for the modules

$$A/(\lambda x^2 - \gamma du, \lambda y^2 - \epsilon du)A, \ (\lambda, \gamma, \epsilon) \in \mathbb{P}^2.$$

- For $(\lambda, \gamma, \epsilon) \in \mathbb{P}^2$, it remains to consider

$$A/(\lambda x^2 - \gamma ud, \lambda y^2 - \epsilon du)A$$

and

$$A/(\lambda x^2 - \gamma du, \lambda y^2 - \epsilon ud)A.$$

Proposition. *In* $Qcoh X$ *we have isomorphisms,*

$$\pi^*(A/(u,x)A) \cong \pi^*(A/(u,y)A)$$

and

$$\pi^*(A/(d,x)) \cong \pi^*(A/(d,y)A).$$

- Are there other isomorphisms?